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ELLIPTICITY CONDITIONS OF THE STATIC EQUATIONS OF NONLINEAR ELASTICITY

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The relations of the nonlinear model of the theory of elasticity are considered. The Cauchy and the strain gradient tensors are taken to be the characteristics of the stress-strain state of a body. Sufficient conditions under which the static equations of elasticity are of elliptic type are established. These conditions are expressed in the form of constraints imposed on the derivatives of the elastic potential with respect to the strain-measure characteristics. The cases of anisotropic and isotropic bodies are treated, including the case where the Almansi tensor is taken to be the strain measure. The plane strain of a body is investigated using actual-state variables.

The system of static equations of the nonlinear theory of elasticity involves equilibrium and continuity equations, constitutive relations, and the expressions of strain measures in terms of displacements. We consider the form of these relations in the Cartesian coordinates of an actual state for the case of plane strains, using the Cauchy stress tensor and the strain gradient tensor as the characteristics of the stress-strain state.

We assume that the deformation from the initial (with Cartesian coordinates of the particles of a body x_k^0) into the actual state (with coordinates x_k) is described by the smooth reversible functions

$$x_k = x_k(x_1^0, x_2^0, x_3^0), \qquad \det\left(\frac{\partial x_k}{\partial x_l^0}\right) \neq 0 \qquad (k, l = 1,\, 2,\, 3),$$

which correspond to the components of displacements u_k defined as the differences between the actual and initial coordinates: $u_k(x_1, x_2, x_3) = x_k - x_k^0(x_1, x_2, x_3)$, where $k = 1$, 2, 3. For plane strains [parallel to the $(x_1,\, x_2)$ plane], the displacement is the two-dimensional vector $(u_1,\, u_2,\, 0)$ whose components depend only on the coordinates of the deformation plane $u_1 = u_1(x_1, x_2)$, $u_2 = u_2(x_1, x_2)$, and $u_3 = 0$. We consider two-dimensional tensors: the strain gradient $C = (C_{\alpha\beta})$ and the displacement gradient $U = (U_{\alpha\beta})$, whose components are defined as the functions of actual coordinates by the formulas [1]

$$C_{\alpha\beta}(x_1, x_2) = \frac{\partial x_\alpha}{\partial x_\beta^0}, \qquad U_{\alpha\beta}(x_1, x_2) = \frac{\partial u_\alpha}{\partial x_\beta}. \tag{1}$$

Hereafter, the Greek subscript takes on the values 1 and 2. We use the tensor C as the strain measure.

The strain gradient can be expressed in terms of the displacement gradient. To this end, we consider the relations

$$x_\alpha = x_\alpha^0 + u_\alpha, \qquad \frac{\partial x_\alpha}{\partial x_\beta^0} = \delta_{\alpha\beta} + \frac{\partial u_\alpha}{\partial x_\sigma}\frac{\partial x_\sigma}{\partial x_\beta^0}$$

(summation over the repeated indices is performed) and rewrite the last relation with allowance for (1) in the component and invariant forms

$$(\delta_{\alpha\sigma} - U_{\alpha\sigma})C_{\sigma\beta} = \delta_{\alpha\beta}, \qquad (\delta - U)C = \delta, \qquad C = (\delta - U)^{-1}, \tag{2}$$

where $\delta = (\delta_{\alpha\beta})$ is the identity tensor. Thus, the strain gradient is a tensor inverse to the difference between the identity tensor and the displacement gradient. The inverse tensor can be expressed in terms of the initial

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tensor and its invariants. Let $H = \delta - U$ be the two-dimensional tensor, and H_1 and H_2 its basis invariants. One can easily see that the Hamilton–Kelly identity for this tensor written in the forms [1]

$$H^2 - H_1 H + H_2\delta = 0, \qquad H\Big(\frac{H_1}{H_2}\,\delta - \frac{1}{H_2}\,H\Big) = \delta,$$

gives the following expression for the inverse tensor H^{-1}:

$$H^{-1} = \frac{H_1}{H_2}\,\delta - \frac{1}{H_2}\,H. \tag{3}$$

Here the invariants H_1 and H_2 are expressed through the basis invariants of the displacement gradient U_1 and U_2 by the formulas

$$H_1 = 2 - U_1, \quad H_2 = 1 - U_1 + U_2, \quad U_1 = U_{11} + U_{22}, \quad U_2 = U_{11}U_{22} - U_{12}U_{21}. \tag{4}$$

It follows from (2)–(4) that the strain gradient is expressed via the displacement gradient by the quasi-linear relation

$$C = \frac{(2 - U_1)\delta - U}{1 - U_1 + U_2}, \qquad C_{\alpha\beta} = \frac{(2 - U_1)\delta_{\alpha\beta} - U_{\alpha\beta}}{1 - U_1 + U_2},$$
$$U_{\alpha\beta} = \frac{\partial u_\alpha}{\partial x_\beta}, \qquad U_1 = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}, \qquad U_2 = \frac{\partial u_1}{\partial x_1}\frac{\partial u_2}{\partial x_2} - \frac{\partial u_1}{\partial x_2}\frac{\partial u_2}{\partial x_1}. \tag{5}$$

The continuity equation in the form of a relation between the density of material ρ and the strain gradient can be obtained from the law of conservation of mass of a body with initial density ρ_0 and arbitrary volume V_0, which deforms into the volume V

$$\int\limits_{V_0} \rho_0 dV_0 = \int\limits_{V} \rho\, dV = \int\limits_{V_0} \rho\Big|\frac{\partial x_\alpha}{\partial x^0_\beta}\Big|\, dV_0$$

in the form of the relation [2] $\rho_0 = \rho|\partial x_\alpha/\partial x^0_\beta|$ or

$$\rho_0 = \rho C_2, \qquad C_2 = C_{11}C_{22} - C_{12}C_{21}, \tag{6}$$

where C_2 is the second basis invariant of the strain gradient.

The stress state of a body in the actual state is characterized by the symmetric Cauchy stress tensor $P = (P_{\alpha\beta})$. We take the constitutive relations in the form of a relation between the Cauchy tensor and the strain gradient. This relation follows from the energy-balance equation applied to an elementary adiabatic process, which corresponds to an arbitrary virtual displacement (δx_β) of the particles in an actual state [3]

$$\frac{1}{\rho}\,P_{\alpha\beta}\,\frac{\partial \delta x_\beta}{\partial x_\alpha} = \delta F,$$

where F is the elastic potential regarded as a function of the components of the strain gradient. Bearing in mind the relations

$$\delta F = \frac{\partial F}{\partial C_{\beta\omega}}\,\delta C_{\beta\omega}, \qquad \delta C_{\beta\omega} = \frac{\partial \delta x_\beta}{\partial x^0_\omega} = C_{\alpha\omega}\,\frac{\partial \delta x_\beta}{\partial x_\alpha},$$

one can write this equation in the form

$$\Big(\frac{1}{\rho}\,P_{\alpha\beta} - C_{\alpha\omega}\,\frac{\partial F}{\partial C_{\beta\omega}}\Big)\frac{\partial \delta x_\beta}{\partial x_\alpha} = 0,$$

from which the constitutive relations

$$P_{\alpha\beta} = \rho C_{\alpha\omega}\,\frac{\partial F}{\partial C_{\beta\omega}} \tag{7}$$

follow by virtue of the arbitrariness of the gradients of virtual displacements.

Finally, setting the principal vector of the body forces (with density f_β) and surface forces (with density $P_{\beta\alpha}n_\alpha$) to zero

$$0 = \int\limits_V \rho f_\beta\, dV + \int\limits_\Sigma P_{\beta\alpha} n_\alpha\, d\sigma = \int\limits_V \Big(\rho f_\beta + \frac{\partial P_{\beta\alpha}}{\partial x_\alpha}\Big)\, dV,$$

we obtain the equilibrium equations of an elastic body of arbitrary volume V and surface Σ [with outward normal (n_α)] in actual variables in the form [1]

$$\rho f_\beta + \frac{\partial P_{\beta\alpha}}{\partial x_\alpha} = 0. \tag{8}$$

System (5)–(8) for the quantities ρ, u_α, $C_{\alpha\beta}$, and $P_{\alpha\beta}$ (ρ_0, F, and f_α are assumed to be specified) is complete and defines the state of equilibrium of an elastic body in plane strain. This system is valid for isotropic and anisotropic bodies, which admit the plane strain.

One can readily see that system (5)–(8) implies second-order equations for displacements. Moreover, first-order equations for the strain gradients, which have the simpler structure, can also be obtained from the system. To this end, it is sufficient to eliminate the density, stresses, and displacements from the system. To eliminate the density and the stresses, we express them in terms of the strain gradient:

$$\rho = \frac{\rho_0}{C_2}, \qquad P_{\alpha\beta} = \frac{\rho_0}{C_2} C_{\alpha\omega} \frac{\partial F}{\partial C_{\beta\omega}}. \tag{9}$$

Elimination of the displacements gives the compatibility equations of the strain gradients

$$\frac{\partial C_{\beta 1}}{\partial x_\sigma} C_{\sigma 2} = \frac{\partial^2 x_\beta}{\partial x_1^0 \partial x_2^0} = \frac{\partial C_{\beta 2}}{\partial x_\sigma} C_{\sigma 1}. \tag{10}$$

The system of four equations (8) and (10) assumes the form

$$\frac{\partial P_{\beta\alpha}}{\partial C_{\sigma\tau}} \frac{\partial C_{\sigma\tau}}{\partial x_\alpha} + \rho f_\beta = 0, \qquad C_{\sigma 2} \frac{\partial C_{\beta 1}}{\partial x_\sigma} - C_{\sigma 1} \frac{\partial C_{\beta 2}}{\partial x_\sigma} = 0, \tag{11}$$

where the stresses and the density are defined by expressions (9), and is the system of first-order equations for strain gradients.

We now investigate the type of system (11). For convenience, we write the system in compact form, introducing the four-dimensional vectors and matrices:

$$A_{kl.\nu} \frac{\partial V_k}{\partial x_\nu} + A_l = 0, \tag{12}$$

where the Latin subscript takes on the values 1, 2, 3, and 4. Moreover, the following notation is introduced:

$$(V_1, V_2, V_3, V_4) = (C_{11}, C_{12}, C_{21}, C_{22}), \qquad (A_1, A_2, A_3, A_4) = (\rho f_1, \rho f_2, 0, 0),$$

$$(A_{kl.1}) = \begin{pmatrix} \frac{\partial P_{11}}{\partial C_{11}} & \frac{\partial P_{12}}{\partial C_{11}} & C_{12} & 0 \\ \frac{\partial P_{11}}{\partial C_{12}} & \frac{\partial P_{12}}{\partial C_{12}} & -C_{11} & 0 \\ \frac{\partial P_{11}}{\partial C_{21}} & \frac{\partial P_{12}}{\partial C_{21}} & 0 & C_{12} \\ \frac{\partial P_{11}}{\partial C_{22}} & \frac{\partial P_{12}}{\partial C_{22}} & 0 & -C_{11} \end{pmatrix}, \quad (A_{kl.2}) = \begin{pmatrix} \frac{\partial P_{21}}{\partial C_{11}} & \frac{\partial P_{22}}{\partial C_{11}} & C_{22} & 0 \\ \frac{\partial P_{21}}{\partial C_{12}} & \frac{\partial P_{22}}{\partial C_{12}} & -C_{21} & 0 \\ \frac{\partial P_{21}}{\partial C_{21}} & \frac{\partial P_{22}}{\partial C_{21}} & 0 & C_{22} \\ \frac{\partial P_{21}}{\partial C_{22}} & \frac{\partial P_{22}}{\partial C_{22}} & 0 & -C_{21} \end{pmatrix}.$$

Let $\varphi(x_1, x_2) = 0$ be the equation of the characteristic curve of system (12). We consider the characteristic matrix (Q_{kl}):

$$(Q_{kl}) = (A_{kl.\nu} n_\nu), \qquad n_\nu = \frac{\partial \varphi}{\partial x_\nu} |\nabla\varphi|^{-1},$$

$$
(Q_{kl}) = \begin{pmatrix}
\frac{\partial P_{\sigma 1}}{\partial C_{11}} n_\sigma & \frac{\partial P_{\sigma 2}}{\partial C_{11}} n_\sigma & C_{\sigma 2} n_\sigma & 0 \\
\frac{\partial P_{\sigma 1}}{\partial C_{12}} n_\sigma & \frac{\partial P_{\sigma 2}}{\partial C_{12}} n_\sigma & -C_{\sigma 1} n_\sigma & 0 \\
\frac{\partial P_{\sigma 1}}{\partial C_{21}} n_\sigma & \frac{\partial P_{\sigma 2}}{\partial C_{21}} n_\sigma & 0 & C_{\sigma 2} n_\sigma \\
\frac{\partial P_{\sigma 1}}{\partial C_{22}} n_\sigma & \frac{\partial P_{\sigma 2}}{\partial C_{22}} n_\sigma & 0 & -C_{\sigma 1} n_\sigma
\end{pmatrix}.
$$

It is well known [4] that system (12) is of elliptic type if the following characteristic equation has no real roots

$$
\det (Q_{kl}) = 0. \tag{13}
$$

Satisfaction of Eq. (13) (its real roots exist) is equivalent to the existence of a nonzero right-hand column vector (d_l) of the characteristic matrix

$$
Q_{kl} d_l = 0. \tag{14}
$$

In this case, introducing the following notation for the components of the eigenvector $d_1 = a_1$, $d_2 = a_2$, $d_3 = b_1$, and $d_4 = b_2$ and expanding system (14), we obtain

$$
\begin{aligned}
Q_{1l} d_l = \frac{\partial P_{\sigma\tau}}{\partial C_{11}} n_\sigma a_\tau + C_{\sigma 2} n_\sigma b_1 = 0, \qquad Q_{2l} d_l = \frac{\partial P_{\sigma\tau}}{\partial C_{12}} n_\sigma a_\tau - C_{\sigma 1} n_\sigma b_1 = 0, \\
Q_{3l} d_l = \frac{\partial P_{\sigma\tau}}{\partial C_{21}} n_\sigma a_\tau + C_{\sigma 2} n_\sigma b_2 = 0, \qquad Q_{4l} d_l = \frac{\partial P_{\sigma\tau}}{\partial C_{22}} n_\sigma a_\tau - C_{\sigma 1} n_\sigma b_2 = 0.
\end{aligned} \tag{15}
$$

Elimination of b_1 and b_2 from (15) gives the homogeneous algebraic system for a_1 and a_2

$$
S_{\lambda\tau} a_\tau = 0, \qquad S_{\lambda\tau} = \frac{\partial P_{\sigma\tau}}{\partial C_{\lambda\mu}} n_\sigma n_\omega C_{\omega\mu}. \tag{16}
$$

The tensor $(S_{\lambda\tau})$ appearing in (16) is symmetric. Indeed, using the expressions for the derivatives of the invariants C_1 and C_2 of the strain gradient with respect to its components [1]

$$
C_1 = \delta_{\sigma\tau} C_{\tau\sigma}, \quad 2C_2 = (\delta_{\sigma\tau} C_{\tau\sigma})^2 - C_{\sigma\tau} C_{\tau\sigma}, \quad \frac{\partial C_1}{\partial C_{\lambda\mu}} = \delta_{\mu\lambda}, \quad \frac{\partial C_2}{\partial C_{\lambda\mu}} = C_1 \delta_{\mu\lambda} - C_{\mu\lambda} \tag{17}
$$

and the representation (3) for the tensor, which is inverse to the strain gradient

$$
C^{-1} = \frac{1}{C_2} (C_1 \delta - C), \qquad C^{-1}_{\mu\lambda} = \frac{1}{C_2} (C_1 \delta_{\mu\lambda} - C_{\mu\lambda}) \qquad [C^{-1}_{\mu\lambda} \equiv (C^{-1})_{\mu\lambda}] \tag{18}
$$

one can express the derivative of density (6) with respect to the strain-gradient components by the formula

$$
\frac{\partial \rho}{\partial C_{\lambda\mu}} = -\frac{\rho_0}{C_2^2} \frac{\partial C_2}{\partial C_{\lambda\mu}} = -\frac{\rho_0}{C_2^2} (C_1 \delta_{\mu\lambda} - C_{\mu\lambda}) = -\rho C^{-1}_{\mu\lambda}. \tag{19}
$$

Reverting to (7) and using (17)–(19), one establishes that the derivatives of stresses with respect to strains are given by the expressions

$$
\frac{\partial P_{\sigma\tau}}{\partial C_{\lambda\mu}} = \rho \frac{\partial F}{\partial C_{\tau\beta}} (\delta_{\mu\beta} \delta_{\lambda\sigma} - C^{-1}_{\mu\lambda} C_{\sigma\beta}) + \rho C_{\sigma\beta} \frac{\partial^2 F}{\partial C_{\tau\beta} \partial C_{\lambda\mu}}. \tag{20}
$$

With allowance for (20), the tensor (16) admits the representation

$$
S_{\lambda\tau} = \rho \frac{\partial F}{\partial C_{\tau\beta}} (n_\lambda n_\omega C_{\omega\beta} - n_\sigma C_{\sigma\beta} n_\omega C_{\omega\mu} C^{-1}_{\mu\lambda}) + \rho C_{\sigma\beta} n_\sigma \frac{\partial^2 F}{\partial C_{\tau\beta} \partial C_{\lambda\mu}} C_{\omega\mu} n_\omega.
$$

By virtue of the relations

$$n_\omega C_{\omega\mu} C^{-1}_{\mu\lambda} = n_\omega \delta_{\omega\lambda} = n_\lambda,$$

$$n_\lambda n_\omega C_{\omega\beta} - n_\sigma C_{\sigma\beta} n_\omega C_{\omega\mu} C^{-1}_{\mu\lambda} = n_\lambda n_\omega C_{\omega\beta} - n_\lambda n_\sigma C_{\sigma\beta} = 0,$$

the first term in this representation vanishes, and the tensor $(S_{\lambda\tau})$ takes the form from which the symmetry of the tensor becomes obvious:

$$S_{\lambda\tau} = \rho n_\sigma C_{\sigma\beta} \frac{\partial^2 F}{\partial C_{\tau\beta} \partial C_{\lambda\mu}} n_\omega C_{\omega\mu} = S_{\tau\lambda}. \tag{21}$$

Under the adopted assumption, system (16) must give the nonzero vector (a_1, a_2) [if $a_1 = a_2 = 0$, Eqs. (15) with $n_\sigma n_\sigma = 1$ and $C_2 \neq 0$ give $b_1 = b_2 = 0$, which contradicts the starting assumptions]; it is, therefore, necessary that $\det(S_{\lambda\tau}) = 0$. Multiplying Eq. (16) by a_λ and performing summation over λ, we infer that, in this case, the quadratic form must be zero $S_{\lambda\tau} a_\tau a_\lambda = 0$.

We now require that, for any vector (a_α), the quadratic form $S_{\lambda\tau} a_\tau a_\lambda > 0$ be positive or negative definite or, in view of (21) and $\rho > 0$,

$$A_{\tau\beta} \frac{\partial^2 F}{\partial C_{\tau\beta} \partial C_{\lambda\mu}} A_{\lambda\mu} > 0 \qquad (A_{\tau\beta} = a_\tau n_\sigma C_{\sigma\beta}). \tag{22}$$

Then, according to the Sylvester conditions [5], $\det(S_{\lambda\tau}) > 0$, and system (16) has the only trivial solution $(a_1, a_2) = (0, 0)$. In this case, Eq. (15) gives only the zero vector $(b_1, b_2) = (0, 0)$. Hence, system (14) has no nonzero eigenvectors (i.e., its determinant does not vanish), which is equivalent to the absence of real roots of the characteristic equation (13). In this case, the quasi-linear system (12) is of elliptic type. Thus, condition (22) [or a condition obtained by changing the sign in the inequality (22)] is a sufficient condition of ellipticity of the system of equations in strains of nonlinear elasticity for plane strains. The ellipticity condition imposes a restriction on the form of the relation between the elastic potential and the deformation characteristics.

For an isotropic material, the relation between the elastic potential and the strain-gradient components is expressed in terms of the invariants of this tensor. We assume that the elastic potential is a function of the form $F(C_1', C_2')$, where $C_1' = \delta_{\sigma\tau} C_{\tau\sigma}$ and $C_2' = C_{\sigma\tau} C_{\tau\sigma}$ are the invariant convolutions of the strain gradient, which are connected with the basis invariants C_1 and C_2 of this tensor by the formulas

$$C_1' = C_1, \qquad C_2' = C_1^2 - 2C_2. \tag{23}$$

It follows from relations (17) and (23) that the derivatives of the convolutions with respect to the strain-gradient components have the values

$$\frac{\partial C_1'}{\partial C_{\lambda\mu}} = \delta_{\mu\lambda}, \qquad \frac{\partial C_2'}{\partial C_{\lambda\mu}} = 2C_{\mu\lambda},$$

which are taken into account to give the second derivatives of the elastic potential with respect to the strain-gradient components in the form

$$\frac{\partial^2 F}{\partial C_{\tau\beta} \partial C_{\lambda\mu}} = 2 \frac{\partial F}{\partial C_2'} \delta_{\beta\lambda} \delta_{\mu\tau} + \frac{\partial^2 F}{\partial C_1'^2} \delta_{\mu\lambda} \delta_{\beta\tau} + 2 \frac{\partial^2 F}{\partial C_1' \partial C_2'} (C_{\mu\lambda} \delta_{\beta\tau} + \delta_{\mu\lambda} C_{\beta\tau}) + 4 \frac{\partial^2 F}{\partial C_2'^2} C_{\mu\lambda} C_{\beta\tau}.$$

In addition to the tensors $(C_{\alpha\beta})$ and $(A_{\alpha\beta})$, we introduce the scalar product $(D_{\alpha\beta}) = (C_{\alpha\sigma} A_{\sigma\beta})$ and consider the invariant convolutions $A_1' = A_1 = A_{\sigma\sigma}$, $A_2' = A_{\sigma\tau} A_{\tau\sigma}$, $D_1' = D_1 = C_{\alpha\sigma} A_{\sigma\alpha}$, and $D_2' = C_{\alpha\sigma} A_{\sigma\beta} C_{\beta\tau} A_{\tau\alpha}$. On the basis of these relations, the ellipticity condition (22) for an isotropic body can be written in the form

$$2 \frac{\partial F}{\partial C_2'} A_2' + \frac{\partial^2 F}{\partial C_1'^2} A_1^2 + 4 \frac{\partial^2 F}{\partial C_1' \partial C_2'} A_1 D_1 + 4 \frac{\partial^2 F}{\partial C_2'^2} D_1^2 > 0. \tag{24}$$

Using the property of the invariant A_2', which follows from (22),

$$A_2' = A_{\sigma\tau} A_{\tau\sigma} = a_\sigma n_\omega C_{\omega\tau} a_\tau n_\lambda C_{\lambda\sigma} = (n_\omega C_{\omega\tau} a_\tau)^2 = A_1^2$$

and determining the components of an arbitrary vector (ξ_α) by the formulas

$$\xi_1 = A_1 = n_\omega C_{\omega\sigma} a_\sigma, \qquad \xi_2 = 2D_1 = 2C_{\alpha\sigma}A_{\sigma\alpha} = 2n_\omega C_{\omega\alpha}C_{\alpha\sigma}a_\sigma,$$

we represent the ellipticity condition of an isotropic body (24) in the form

$$2\,\frac{\partial F}{\partial C_2'}\,\xi_1^2 + \frac{\partial^2 F}{\partial C_\alpha' \partial C_\beta'}\,\xi_\alpha \xi_\beta > 0. \tag{25}$$

One can see from (25) that the ellipticity condition is satisfied if the first term is positive and the quadratic form is positive definite, i.e., the derivatives of the elastic potential with respect to the invariant convolutions satisfy the inequalities [5]

$$\frac{\partial F}{\partial C_2'} > 0, \qquad \frac{\partial^2 F}{\partial C_1'^2} > 0, \qquad \frac{\partial^2 F}{\partial C_1'^2}\,\frac{\partial^2 F}{\partial C_2'^2} - \Big(\frac{\partial^2 F}{\partial C_1' \partial C_2'}\Big)^2 > 0. \tag{26}$$

Inequalities (26) are the sufficient conditions of ellipticity of the equations of nonlinear elasticity for an isotropic body in plane strain.

To investigate the finite strains of an elastic body in actual variables, the Almansi tensor $\varepsilon = (\varepsilon_{\alpha\beta})$, whose components are given by the expressions

$$2\varepsilon_{\alpha\beta}(x_1, x_2) = \delta_{\alpha\beta} - \frac{\partial x_\sigma^0}{\partial x_\alpha}\,\frac{\partial x_\sigma^0}{\partial x_\beta},$$

is also used.

The tensor ε is a function of the strain gradient C and the corresponding conjugate tensor C^* [1]:

$$2\varepsilon = \delta - {C^*}^{-1} C^{-1} = \delta - (CC^*)^{-1}.$$

We also consider the strain tensor $B = (B_{\alpha\beta})$ to be the isotropic function of the tensor ε, which is connected to the tensors C and C^* by simpler relations than the tensor ε:

$$B = (\delta - 2\varepsilon)^{-1} = CC^*, \qquad B_{\alpha\beta} = C_{\alpha\sigma}C^*_{\sigma\beta} = C_{\alpha\sigma}C_{\beta\sigma}.$$

The invariant convolutions B_1' and B_2' of the tensor B are expressed through the tensor components $(C_{\sigma\tau})$ by the formulas

$$B_1' = B_{\alpha\alpha} = C_{\alpha\sigma}C_{\alpha\sigma}, \qquad B_2' = B_{\alpha\beta}B_{\beta\alpha} = C_{\alpha\sigma}C_{\beta\sigma}C_{\beta\tau}C_{\alpha\tau}.$$

Using the representation (3) for the inverse tensor, one can readily establish that B and ε are connected by the relations

$$B = \frac{(1 - 2\varepsilon_1)\delta - 2\varepsilon}{1 - 2\varepsilon_1 + 4\varepsilon_2}, \qquad \varepsilon = \frac{(B_2 - B_1)\delta + B}{2B_2}, \tag{27}$$

where ε_1, ε_2, B_1, and B_2 are the basis invariants of the tensors ε and B. From (27), one obtains the equalities

$$\Big(1 - 2\varepsilon_1 + 4\varepsilon_2 - \frac{1}{B_2}\Big)B - \Big(2 - 2\varepsilon_1 - \frac{B_1}{B_2}\Big)\delta \equiv 0,$$
$$1 - 2\varepsilon_1 + 4\varepsilon_2 = \frac{1}{B_2}, \qquad 2(1 - \varepsilon_1) = \frac{B_1}{B_2}. \tag{28}$$

Assuming the relative density of the material to be a finite nonzero quantity and using the continuity equation written in ε and B

$$\Big(\frac{\rho}{\rho_0}\Big)^2 = 1 - 2\varepsilon_1 + 4\varepsilon_2 = \frac{1}{B_2} \neq 0, \infty,$$

we have $1 - 2\varepsilon_1 + 4\varepsilon_2 \neq 0$ and $B_2 \neq 0$, which, in accordance with (28), allows us to establish the following relations between the invariant convolutions of the tensors ε and B:

$$\varepsilon_1' = \frac{B_1'(B_1'-1)-B_2'}{B_1'^2-B_2'}, \qquad \varepsilon_2' = \frac{2B_2'+(B_1'^2-B_2')(B_1'^2-2B_1'-B_2')}{2(B_1'^2-B_2')^2}. \tag{29}$$

By virtue of the relation between the strain measures considered above, we can assume that the elastic potential in the ellipticity condition (22) depends on the strain-gradient components by means of the components of the Almansi tensor: $F[\varepsilon_{\alpha\beta}(C_{\sigma\tau})]$. In the case of an isotropic body, this relation is expressed in terms of the basis invariants of the Almansi tensor. Using the invariants ε_1' and ε_2' or, by virtue of (29), the invariants B_1' and B_2', which are connected with $C_{\sigma\tau}$ by relatively simple relations as basis invariants, we have $F(B_1'(C_{\sigma\tau}), B_2'(C_{\sigma\tau}))$.

Using the formulas

$$\frac{\partial^2 F}{\partial C_{\tau\beta}\partial C_{\lambda\mu}} = \frac{\partial F}{\partial B_1'}\frac{\partial^2 B_1'}{\partial C_{\tau\beta}\partial C_{\lambda\mu}} + \frac{\partial F}{\partial B_2'}\frac{\partial^2 B_2'}{\partial C_{\tau\beta}\partial C_{\lambda\mu}}$$

$$+\frac{\partial^2 F}{\partial B_1'^2}\frac{\partial B_1'}{\partial C_{\tau\beta}}\frac{\partial B_1'}{\partial C_{\lambda\mu}} + \frac{\partial^2 F}{\partial B_1'\partial B_2'}\left(\frac{\partial B_1'}{\partial C_{\tau\beta}}\frac{\partial B_2'}{\partial C_{\lambda\mu}} + \frac{\partial B_2'}{\partial C_{\tau\beta}}\frac{\partial B_1'}{\partial C_{\lambda\mu}}\right) + \frac{\partial^2 F}{\partial B_2'^2}\frac{\partial B_2'}{\partial C_{\tau\beta}}\frac{\partial B_2'}{\partial C_{\lambda\mu}},$$

$$\frac{\partial B_1'}{\partial C_{\tau\beta}} = 2C_{\tau\beta}, \quad \frac{\partial B_2}{\partial C_{\tau\beta}} = 4S_{\tau\beta}, \quad S_{\tau\beta} = C_{\tau\sigma}C_{\alpha\sigma}C_{\alpha\beta},$$

$$\frac{\partial^2 B_1'}{\partial C_{\tau\beta}\partial C_{\lambda\mu}} = 2\delta_{\tau\lambda}\delta_{\beta\mu}, \quad \frac{\partial^2 B_2'}{\partial C_{\tau\beta}\partial C_{\lambda\mu}} = 4(\delta_{\mu\beta}C_{\lambda\sigma}C_{\tau\sigma} + \delta_{\lambda\tau}C_{\alpha\beta}C_{\alpha\mu} + C_{\lambda\beta}C_{\tau\mu}),$$

the ellipticity condition (22) for an isotropic body assumes the form

$$a_\tau m_\beta \frac{\partial^2 F}{\partial C_{\tau\beta}\partial C_{\lambda\mu}} a_\lambda m_\mu = \frac{\partial^2 F}{\partial B_\alpha' \partial B_\beta'}\eta_\alpha\eta_\beta + 2\frac{\partial F}{\partial B_1'}(a_\sigma a_\sigma)(m_\beta m_\beta)$$

$$+\frac{\partial F}{\partial B_2'}[(m_\beta m_\beta)(l_\sigma l_\sigma) + (a_\sigma a_\sigma)(k_\alpha k_\alpha) + \eta_1^2] > 0, \tag{30}$$

where $m_\beta = n_\omega C_{\omega\beta}$, $l_\sigma = 2a_\tau C_{\tau\sigma}$, $k_\alpha = 2C_{\alpha\beta}m_\beta$, $\eta_1 = 2a_\sigma C_{\sigma\beta}m_\beta$, and $\eta_2 = 4a_\sigma S_{\sigma\beta}m_\beta$.

It can easily be seen that, for an arbitrary nonzero vector (a_1, a_2), the following forms are positive definite: $m_\beta m_\beta a_\sigma a_\sigma > 0$, and $m_\beta m_\beta l_\sigma l_\sigma + k_\alpha k_\alpha a_\sigma a_\sigma + \eta_1^2 > 0$; consequently, condition (30) can be satisfied, provided the first derivatives of the elastic potential are positive and the quadratic form is positive definite:

$$\frac{\partial F}{\partial B_1'} > 0, \quad \frac{\partial F}{\partial B_2'} > 0, \quad \frac{\partial^2 F}{\partial B_1'^2} > 0, \quad \frac{\partial^2 F}{\partial B_1'^2}\frac{\partial^2 F}{\partial B_2'^2} - \left(\frac{\partial^2 F}{\partial B_1'\partial B_2'}\right)^2 > 0. \tag{31}$$

Inequalities (31) represent another [different from (26)] form of the sufficient ellipticity conditions for the static equations of isotropic nonlinear elasticity for plane strains. These conditions restrict the values of the first and second derivatives in the functional relation between the elastic potential of the form $F(B_1', B_2')$ and the basis strain invariants.

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